The Duality between IIB String Theory on PP-wave and $\mathcal{N} = 4$ SYM: a Status Report.

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Plan of the talk

Introduce the basic concepts:

- The maximally supersymmetric IIB PP–wave solution (Blau et al.)
  - Its relation with $AdS_5 \times S^5$ (Blau et al.)
  - The consequences implied by the AdS/CFT duality: the BMN proposal (Berenstein, Maldacena, and Nastase)
- Light–cone quantization of IIB superstring on this background (Metsaev)
- The duality between IIB strings on the PP-wave and subsector of $\mathcal{N} = 4$ SYM (BMN, Metsaev and Tseytlin, Kristjansen et al., Constable et al., . . .):
  - The matching of the parameters
  - Dictionary between string states and Gauge Theory operators v1
  - The matching of quantum numbers

Further developments

- GT operator mixing: string/GT dictionary v2

- Describing the 3–string interaction. Two possible results

A_s) Smoothly connected to flat space results
B_s) Use the intuition derived from the study of AdS/CFT.

- Tracing the string interaction on the gauge theory side.

A_g) Use data extracted from GT correlators (Constable et al., Dobashi et al. . . .)

B_g) Match against the anomalous dimension matrix in a new GT basis v3 (Gross et al., Gomis et al., . . .)

- Conclusions: what are the lessons we can learn from the PP–wave limit for the full AdS/CFT duality ?

The PP-wave background and its symmetries

- A constant dilaton

- A constant null 5–form: $F_{+1234} = F_{+5678} = 2\mu$

- The metric: $ds^2 = -4dx^+dx^- - \mu^2 \sum_{I=1}^8 x_I x^I (dx^+)^2 + \sum_{I=1}^8 dx_I dx^I$

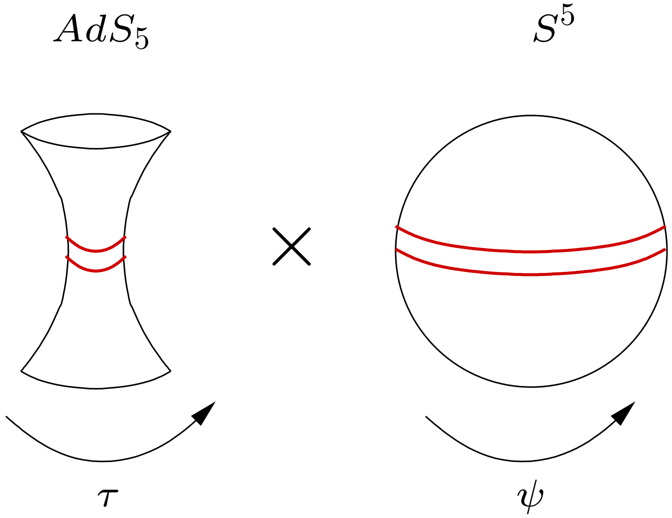
The key features:

- It's a curved background ($R_{++} \sim \mu$)

- It has 30 bosonic symmetries (as $AdS_5 \times S_5$!) and a Z_2 discrete symmetry : $SO(4) \times SO(4) \times H(8) \times R^+ \times Z_2$

- It preserves 32 supersymmetries !

The relation with $AdS_5 \times S^5$



Penrose limit: focus on the region between the red lines. The result is the maximally supersymmetric PP–wave.

- Many features of the PP-wave background can be derived from this limiting procedure (like symmetries, the algrebra).
- This limit can be applied to any spacetime to generate plane-waves
- Clear semi-classical interpretation: point-like strings spinning along S^5.

- Start from the $AdS_5 \times S^5$ metric in global coordinates
  - $ds^2 = -\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\psi^2 + d\theta^2 + \sinh^2\theta\, d\Omega_3'^2$
  - The dictionary between string and GT parameters is

$$4\pi g_s = g_{YM}^2 \quad , \quad \frac{R^2}{\alpha'} = \sqrt{g_t} \quad , \quad g_t \equiv g_{YM}^2 N_c$$

- Introduce the light-cone coordinates

$$x^+ = \frac{\tau + \psi}{2\mu} \; , \; x^- = R^2\mu\frac{\tau - \psi}{2} \; , \; r = R\rho \; , \; y = R\theta \; .$$

and take the $R \to \infty$, by keeping $x^{\pm}$, r and y fixed.

- Everything can be generalized to supergravity (Gueven)

The basic relations

- Relation between string and SYM symmetry generators

$$\boxed{\frac{1}{\mu}\frac{\partial}{\partial x^{+}} = \frac{H}{\mu} = \Delta - J \ , \quad 2\mu p^{+} = \frac{\Delta + J}{R^2} \to \frac{2J}{R^2}}$$

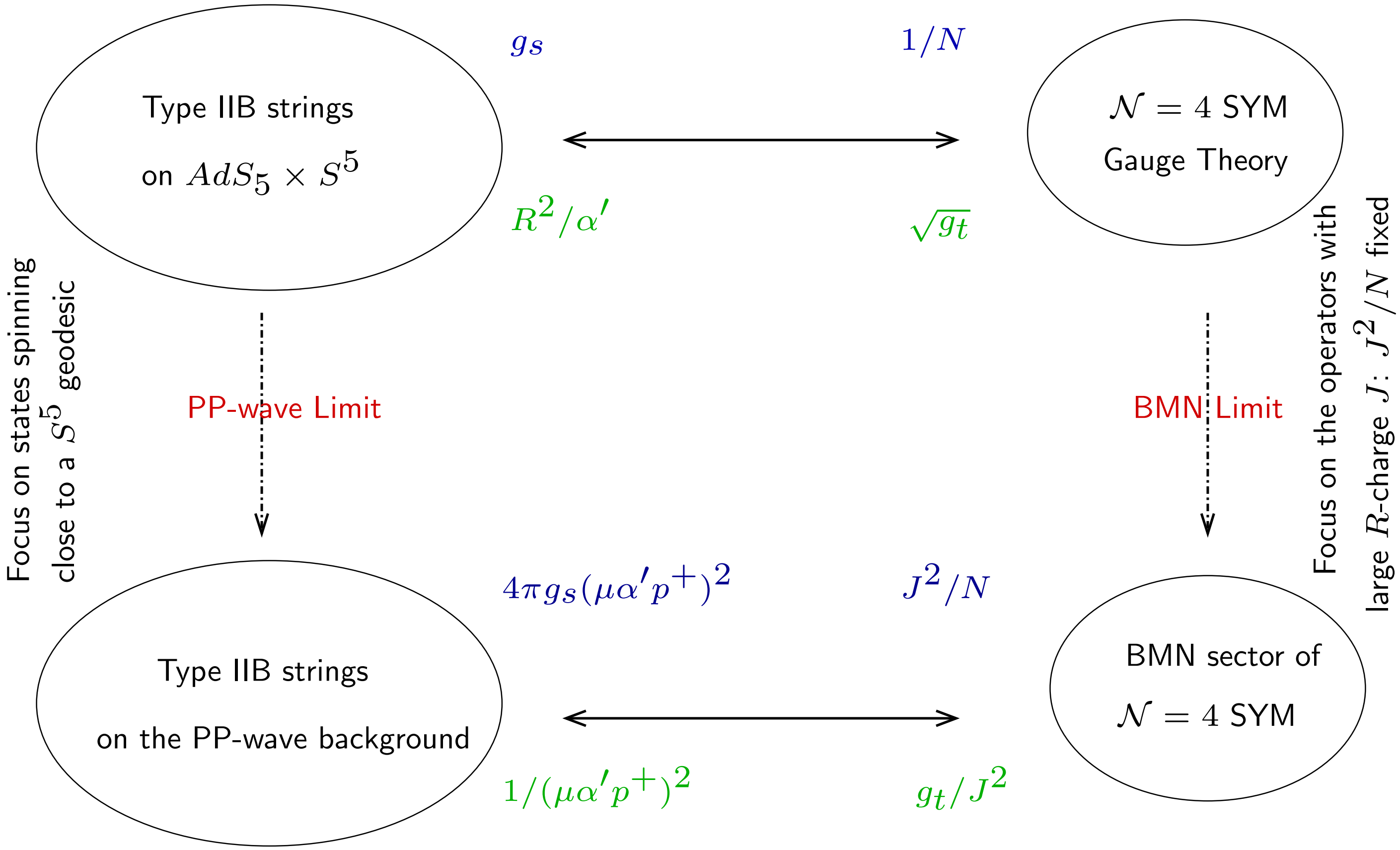
- The relevant parameters are

$$\boxed{\frac{1}{(\mu\alpha' p^{+})^2} = \frac{g_t}{J^2} \equiv \lambda' \ , \quad 4\pi g_s (\mu\alpha' p^{+})^2 = \frac{J^2}{N_c} \equiv g_2}$$

- On the GT side the Penrose limit requires:

$$\boxed{\Delta \to \infty \ , \ J \to \infty \ , \ N_c \to \infty \ \text{with} \ \frac{J}{\sqrt{N_c}} \ , \ \frac{\Delta}{\sqrt{N_c}} \ , \ g_{YM}^2 \ fixed}$$

From AdS/CFT to the BMN duality



Type IIB strings

on $AdS_5 \times S^5$

g_s

R^2/α'

$1/N$

$\sqrt{g_t}$

$\mathcal{N} = 4$ SYM

Gauge Theory

Focus on states spinning close to a S^5 geodesic

PP-wave Limit

BMN Limit

Focus on the operators with large R-charge J: J^2/N fixed

Type IIB strings

on the PP-wave background

$4\pi g_s(\mu\alpha' p^+)^2$

$1/(\mu\alpha' p^+)^2$

J^2/N

g_t/J^2

BMN sector of

$\mathcal{N} = 4$ SYM

The free string I

The world–sheet action is particularly simple in the light cone gauge. In the bosonic sector: $X^+(\tau,\sigma) = e(\alpha)\tau$, with $\alpha = \alpha' p^+$

$$S_b = \frac{1}{4\pi\alpha'}\int d\tau \int\limits_0^{2\pi|\alpha|} d\sigma \left[(\partial_\tau X)^2 - (\partial_\sigma X)^2 - \mu^2 X^2\right]$$

In the fermionic sector: $\theta^1(\tau,\sigma)$ and $\theta^2(\tau,\sigma)$ are 10D chiral spinor satisfying $\left(\Gamma^0 + \Gamma^9\right)\theta^i = 0$. The action is

$$S_f = \frac{1}{4\pi\alpha'}\int d\tau \int\limits_0^{2\pi|\alpha|} d\sigma \; i\left\{e(\alpha)\left[\theta^1\partial_+\theta^1 + \theta^2\partial_-\theta^2\right] - 2\mu\theta^1\Pi\theta^2\right\}$$

The string dynamics is described by free 2D action: 8 massive bosons and 8 massive fermions $\left[\Pi = \Gamma^1 \ldots \Gamma^4 \rightarrow (1_{4\times4}, -1_{4\times4})\right]$.

The free string II

$$X(\tau,\sigma) = i\sqrt{\frac{\alpha'}{2}}\Big[\frac{a_0}{\sqrt{\omega_0}}\mathrm{e}^{-i\mu\tau} - \frac{a_0^\dagger}{\sqrt{\omega_0}}\mathrm{e}^{i\mu\tau} + \sum_{n=1}^{\infty}$$
$$\Big(\frac{a_n^1}{\sqrt{\omega_n}}\mathrm{e}^{-i\frac{\omega_n\tau-n\sigma}{|\alpha|}} - \frac{a_n^{1\dagger}}{\sqrt{\omega_n}}\mathrm{e}^{i\frac{\omega_n\tau-n\sigma}{|\alpha|}} + \frac{a_n^2}{\sqrt{\omega_n}}\mathrm{e}^{-i\frac{\omega_n\tau+n\sigma}{|\alpha|}} - \frac{a_n^{2\dagger}}{\sqrt{\omega_n}}\mathrm{e}^{i\frac{\omega_n\tau+n\sigma}{|\alpha|}}\Big)\Big]$$

- Each mode is a harmonic oscillators, contributing $\omega_n = \sqrt{n^2+(\alpha\mu)^2}$ to the light-cone Hamiltonian:

$$H_2 = \frac{1}{\alpha}\left[\frac{\omega_0}{2}(a_0^\dagger a_0 + a_0 a_0^\dagger) + \sum_{i=1}^{2}\sum_{n=1}^{\infty}\frac{\omega_n}{2}(a_n^{i\dagger}a_n^i + a_n^i a_n^{i\dagger})\right]$$

- a^1 (a^2) becomes a left (right) moving mode in the $\mu \to 0$ limit.

The fermionic mode expansion is

$$\sqrt{\frac{|\alpha|}{\alpha'}}\theta^1 = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-i\mu\tau}\theta_0 + \mathrm{e}^{i\mu\tau}\theta_0^\dagger\right) + \sum_{n=1}^{\infty} c_n \left[\mathrm{e}^{-i\frac{\omega_n\tau - n\sigma}{|\alpha|}}\theta_n^1 + \mathrm{e}^{i\frac{\omega_n\tau - n\sigma}{|\alpha|}}(\theta_n^1)^\dagger\right] +$$

$$+ i\sum_{n=1}^{\infty} c_n \frac{\omega_n - n}{\alpha\mu}\left[\mathrm{e}^{-i\frac{\omega_n\tau + n\sigma}{|\alpha|}}\Pi\theta_n^2 - \mathrm{e}^{i\frac{\omega_n\tau + n\sigma}{|\alpha|}}\Pi(\theta_n^2)^\dagger\right]$$

$$\sqrt{\frac{|\alpha|}{\alpha'}}\theta^2 = \frac{e(\alpha)}{\sqrt{2}}\left(i\mathrm{e}^{i\mu\tau}\Pi\theta_0^\dagger - i\mathrm{e}^{-i\mu\tau}\Pi\theta_0\right) + \sum_{n=1}^{\infty} c_n \left[\mathrm{e}^{-i\frac{\omega_n\tau + n\sigma}{|\alpha|}}\theta_n^2 + \mathrm{e}^{i\frac{\omega_n\tau + n\sigma}{|\alpha|}}(\theta_n^2)^\dagger\right] +$$

$$- i\sum_{n=1}^{\infty} c_n \frac{\omega_n - n}{\alpha\mu}\left[\mathrm{e}^{-i\frac{\omega_n\tau - n\sigma}{|\alpha|}}\Pi\theta_n^1 - \mathrm{e}^{i\frac{\omega_n\tau - n\sigma}{|\alpha|}}\Pi(\theta_n^1)^\dagger\right]$$

with $c_n = \frac{1}{\sqrt{1+\rho_n^2}} \quad , \quad \rho_n = \frac{\omega_n - n}{\mu\alpha}$

The free spectrum (at fixed p^+)

- There is a unique state of zero energy: $|v\rangle$, $a_n^i|v\rangle = \theta_n^i|v\rangle = 0 \ \forall\, i, n$.

- The spectrum is built with the creation operators and is discrete.

- The states containing only $\theta_0^\dagger$ fill out a short (half-BPS) supermultiplet.

- $|v\rangle \leftrightarrow \sum_5^8 g_{ii} + iC_{5678}$ and $|0\rangle = \theta_0^{5\dagger} \ldots \theta_0^{8\dagger}|v\rangle \leftrightarrow \chi + i\mathrm{e}^{i\phi}$.

- $|v\rangle$ and $|0\rangle$ have opposite Z_2 parity.

- The stringy (non–BPS) states must satisfy the level matching condition:

$$\sum_n n\,(a_n^{1\dagger}a_n^1 + \theta_n^{1\dagger}\theta_n^1) = \sum_n n\,(a_n^{2\dagger}a_n^2 + \theta_n^{2\dagger}\theta_n^2)\ .$$

The GT side

The $\mathcal{N}=4$ Lagrangian.

- The quadratic part in the $\mathcal{N}=2$ language.

$$\mathcal{L}=\mathrm{Tr}\left[\tfrac{1}{4}F^{mn}F_{mn}+D_m\bar{Z}D^mZ+\tfrac{1}{2}D_m\phi^{i'}D^m\phi^{i'}+i\bar{\lambda}^uD_m\bar{\sigma}^m\lambda_u+i\bar{\psi}_{\dot{u}}D_m\bar{\sigma}^m\psi^{\dot{u}}\right]$$

- The relevant group symmetry is $SO(4)\times SO(4)\subset SO(2,4)\times SO(6)$ (Δ and J play a special role!).

- The bosonic part in the $\mathcal{N}=1$ language. Notice the F-terms ($i=1,2,3$)

$$L=\mathrm{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+D_\mu\bar{\phi}_iD^\mu\phi_i+g^2[\bar{\phi}_i,\bar{\phi}_j][\phi_i,\phi_j]-\frac{g^2}{2}\Big(\sum_i[\bar{\phi}_i,\phi_i]\Big)^2\Big]$$

BMN dictionary

- Identify the two $SO(4)$ in the bulk as the Euclidean rotations and the residual R-symmetry on the SYM side

- Consider gauge invariant composite operators (traces) with many Z insertions ($J \to \infty$) and few (a finite number) impurities.

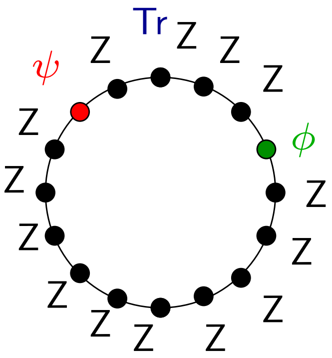
- The impurities allowed are: $\left\{ \begin{array}{ll} \text{Bosonic} & \phi^i \; , \; D_i Z \\ \text{Fermionic} & \lambda^{\alpha}_{u} \; , \; \bar{\psi}^{\dot{\alpha}}_{\dot{u}} \end{array} \right.$

- Examples for supergravity modes

$$|v\rangle_\alpha \quad \leftrightarrow \quad \frac{1}{\sqrt{JN^J}}\mathrm{Tr}[Z^J] \; ,$$

$$(\hat{a}_0^{\phi^1})^\dagger |v\rangle_\alpha \quad \leftrightarrow \quad \frac{1}{\sqrt{N^{J+1}}}\mathrm{Tr}[\phi^1 Z^J] \; ,$$

Gauge theory operators provides a discretize model for closed strings



● Z insertion

● Impurity

● Impurity

Supergravity modes correspond to the usual CPO's

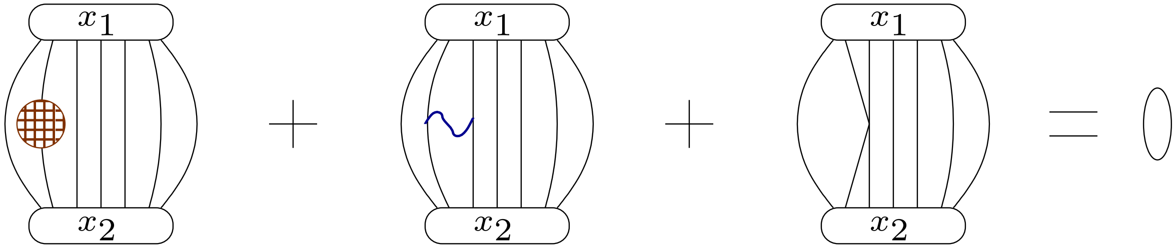
$$(\hat{a}_0^{\phi^2})^\dagger(\hat{a}_0^{\phi^1})^\dagger|v\rangle_\alpha \quad \leftrightarrow \quad \frac{1}{\sqrt{N^{J+2}(J+2)}}\sum_{\hat{l}=0}^{J}\mathrm{Tr}[\phi^1 Z^{\hat{l}}\phi^2 Z^{(J-\hat{l})}]\ .$$

For string modes, use the the X mode expansion to derive the phase!

$$(\hat{a}_m^{\phi^2})^\dagger(\hat{a}_{-m}^{\phi^1})^\dagger|v\rangle_\alpha \leftrightarrow \overbrace{\frac{1}{\sqrt{N^{J+2}(J+2)}}\sum_{\hat{l}=0}^{J}\mathrm{Tr}[\phi^1 Z^{\hat{l}}\phi^2 Z^{(J-\hat{l})}]\mathrm{e}^{2\pi\mathrm{i}\frac{(\hat{l}+1)m}{J+2}}}^{O_{12}^J}$$

Matching the anomalous dimensions

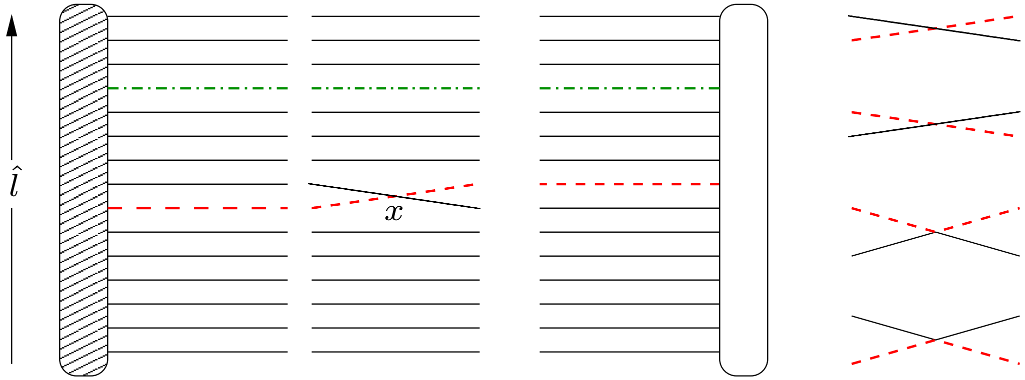
- A useful cancellation in $\mathcal{N}=4$ for CPO's $(x_1 \neq x_2)$



Scalar self-energy Gluon exchange D-term interaction

- The scalar lines are not differentiated: these interaction are flavor blind!
- The presence of the phases does not spoil the cancellation (the impurities are not displaced).
- Only the F-terms contribute to the 1-loop anomalous dimension!

- Let's visualize the action of an F-term insertion



$+\mathrm{Tr}[\phi Z \bar{\phi} \bar{Z}]$ (A)

$+\mathrm{Tr}[\phi \bar{Z} \bar{\phi} Z]$ (B)

$-\mathrm{Tr}[\phi \bar{\phi} \bar{Z} Z]$ (C)

$-\mathrm{Tr}[\phi Z \bar{Z} \bar{\phi}]$ (D)

- The space-time dependence of (A)–(D) is the same.

- $$\text{(A)}\ O^J_{12} \to g_t \left(\mathrm{e}^{-2\pi i \frac{m}{J+2}}\right) O^J_{12} \qquad \text{(B)}\ O^J_{12} \to g_t \left(\mathrm{e}^{2\pi i \frac{m}{J+2}}\right) O^J_{12}$$
  $$\text{(C)}\ O^J_{12} \to -g_t\, O^J_{12} \qquad \text{(D)}\ O^J_{12} \to -g_t\, O^J_{12}$$

- (A) + (B) + (C) + (D) $O^J_{12} \to g_t \left(\mathrm{e}^{2\pi i \frac{m}{J+2}} - \mathrm{e}^{-2\pi i \frac{m}{J+2}} - 2\right) O^J_{12}$

Now one can easily compute the 2-point function

$$\langle \overline{O}^J_{12}(x) O^J_{12}(0) \rangle = 1 - \delta \log x^2 + \dots$$

and derive the 1-loop anomalous dimension δ

$$\boxed{\delta = m^2 \lambda'}$$

This agrees with the string formula!

$$\frac{H_2}{\mu} (\hat{a}^{\phi^2}_m)^\dagger (\hat{a}^{\phi^1}_{-m})^\dagger |v\rangle_\alpha = \frac{2\omega_m}{\mu\alpha} (\hat{a}^{\phi^2}_m)^\dagger (\hat{a}^{\phi^1}_{-m})^\dagger |v\rangle_\alpha$$

and when $\lambda' = (\mu\alpha)^2 \to 0$

$$\frac{2\,\omega_m}{\mu\alpha} = 2\sqrt{1 + \frac{m^2}{(\mu\alpha)^2}} \to 2 - m^2\lambda'$$

Open problems

- Some conceptual and technical issues:

1 Is the dictionary just presented really correct ?

  - Mixing with multi-trace operators (all powers of g_2)
  - Higher genus contributions (even powers of g_2)

2 Construct the 3-string interaction C_{3s}

3 What is the gauge theory quantity corresponding to C_{3s} ?

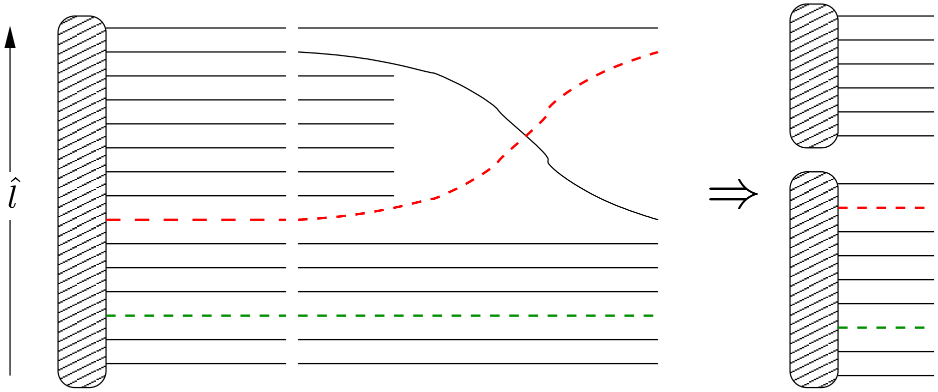
Opt. A The anomalous dimension matrix in a particular operator basis including also multitrace operators (Gross et al., Gomis et al., . . .).

Opt. B Information on the 3-point correlation function among single trace operators (Constable et al., Dobashi et al. , . . .)

GT computations revised: the operator mixing

- In the computation of the anomalous dimension we considered only between two consecutive fields

- Interactions involving two distant fields are suppressed by $1/N$ in the usual AdS/CFT and by g_2 in the BMN limit



This contraction of a F-term splits the trace in two distinct parts. Notice that this is a non-planar, quantum effect (it's of order $g_2\lambda'$).

- The phase factor is unmodified by the above process (but ...!)

The idea of Beisert et al. is to treat (in the computation of anomalous dimensions) the interactions between consecutive fields as a free Hamiltonian and the interactions between distant fields as a perturbation.

- Apply QM non-degenerate perturbation theory. At first order
  - There is no "energy shift" ("energy" ↔ anomalous dimension)
  - But the form of eigenvalues is modified ($r = J'/J$)

$$O'^{J}_{12,m} = O^{J}_{12,m} - \sum_{k,r} \frac{g_2\, r^{3/2}\sqrt{1-r}\, \sin^2(\pi m r)\, k}{\sqrt{J}\,\pi^2 (k-mr)^2 (k+mr)} O^{J'}_{12,k} \mathrm{Tr}[Z^{J-J'}]$$

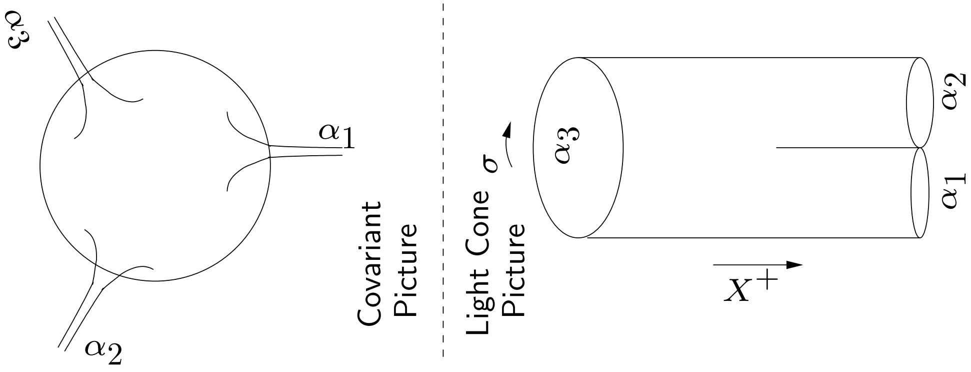
- Notice that there is no λ'! In $\sum_{m\neq n}\langle m|h_+|n\rangle/(E^{(0)}_m - E^{(0)}_n)$, it's present both at the numerator and at the denominator.
- First order non-degenerate perturbation theory is ok (a non-trivial check is necessary)

- Proceed and apply second order non-degenerate perturbation theory
- The result for the 2-loop anomalous dimension is

$$\delta_2 = \lambda' g_2^2 \left(\frac{1}{12} + \frac{35}{32\pi^2 m^2} \right)$$

- However one has to take this formula with some caution: second order non-degenerate perturbation theory breaks down
- At this order the single/triple trace mixing is relevant. There are states of degenerate energy that spoil the validity of the eigenvectors formula
- The main question: what is the role of the eigenstates $O'^{J}_{12.m}$ in the BMN duality ? what is their string dual ?

The interaction vertex



The goal is to derive the coupling C_{3s} among 3 strings.

C_{3s} is a function of the quantum numbers of the external states.

The general approach is in two steps:

- Introduce an independent Hilbert space for each external state
- The interaction is described by a generating functional: a ket-state $|H_3\rangle$ in the tensor product of the 3 Hilbert spaces

$$C_{3s} = (\langle 1| \otimes \langle 2| \otimes \langle 3|)\ |H_3\rangle$$

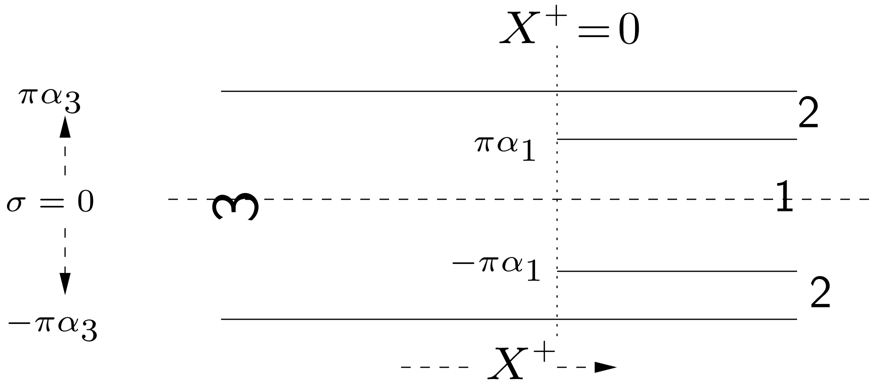
The usual approach is to fix $|H_3\rangle$ by imposing the PP-wave symmetries

I present a different derivation: use the standard path integral techniques

$$U_{t'}^{t''} = \int \prod_i \frac{da^*(t_i)da(t_i)}{2\pi i} \exp\left\{\frac{1}{2}(a^*(t'')a(t'')+a^*(t')a(t'))\right\}$$

$$\times \quad \exp\left\{ i\int_{t',a}^{t'',a^*}\left(\frac{1}{2i}(\dot{a}^*(t)a(t)-a^*(t)\dot{a}(t)-h(a^*(t),a(t))\right)dt\right\}$$

Apply this general formula to our case



A world–sheet picture of the 3-string vertex

- The dynamics governed by the classical e.o.m., except in $X^+ = 0$.

- Impose continuity of phase space variables at $X^+ = 0$.
  - These constraints yields some relations among the modes
  - Choose a set of independent variables (a_n^*).
  - Solve the linear system: $a_m = f_m(a_n^*)$.

- Rewrite $U_{t'}^{t''}$ using only the a_n^*'s.

- In terms of the oscillators $a_n = \frac{1}{\sqrt{2}}(a_n^1 + a_n^2)$, $a_{-n} = \frac{i}{\sqrt{2}}(a_n^1 - a_n^2)$

$$\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n,m=-\infty}^{\infty} a_n^{(r)*} N_{nm}^{rs} a_n^{(s)*}\right] = {}_{123}\langle v|\mathrm{e}^{a_n^{(r)*}\hat{a}_n^{(r)}}|H_3\rangle_{bos}$$

with an "explicit" expression for N_{nm}^{rs}

The Neumann Matrices

- The first ingredients μ-independent $(C_{nm} = n\delta_{nm},\ \beta = \alpha_1/\alpha_3)$

$$C^{1/2}\ A^{(1)}_{nm}\ C^{-1/2} = \int_{-\pi\alpha_1}^{\pi\alpha_1} \frac{d\sigma}{\pi\alpha_1} \cos\frac{m\sigma}{|\alpha_1|} \cos\frac{n\sigma}{|\alpha_3|}$$

$$= \quad \frac{1}{\pi}\frac{(-1)^m n\beta}{n^2\beta^2 - m^2}\sin(\pi n\beta) \equiv X^{(1)}_{nm}, \quad n, m > 0$$

- An important combination is Γ_a (μ-dependent)

$$\Gamma_a = \sum_{r=1}^{3} X^{(r)} C_{(r)} X^{(r)T}, \qquad \text{where} \qquad \left[C_{(r)}\right]_{mn} = \omega_{m(r)}\delta_{mn}$$

$$\boxed{\overline{N}^{rs}_{mn} = \delta^{rs}\delta_{mn} - 2\big(C^{1/2}_{(r)} X^{(r)T}\Gamma_a^{-1} X^{(s)} C^{1/2}_{(s)}\big)_{mn}}$$

- This expression is "explicit" because it is hard to invert Γ_a!
- The leading ($\mu \to \infty$) term is simple $(\Gamma_a)_{mn} = 2\mu|\alpha_3|\delta_{mn}$ (Huang)
- No analytic expression is known for $\mu \neq 0$. The large μ behaviour has been thoroughly studied (Chu et al., He et al.)
- From He et al. ($s_{1m} = s_{2m} = 1,\ s_{3m} = -2\sin(\pi m y)$ and $n, m > 0$)

$$N^{rs}_{mn} \approx \frac{1}{2\pi}\frac{(-1)^{r(m+1)+s(n+1)}}{\alpha_s\omega_{rm} + \alpha_r\omega_{sn}}\sqrt{\frac{|\alpha_r\alpha_s|(\omega_{rm} + \mu\alpha_r)(\omega_{sn} + \mu\alpha_s)}{\omega_{rm}\omega_{sn}}}s_{rm}s_{sn}$$

The fermionic sector

- Apply the same path integral approach to derive the fermionic exponential

- The exponential part of $|H_3\rangle$ (indicated as $|V\rangle$) automatically satisfies all the kinematical symmetries

- Two related problems:
  - One still needs to implement the dynamical symmetries (why ?)
  - Ambiguity related to the boundary conditions: $|V\rangle$ is not unique. At least two solutions:

    $|V\rangle = \exp\{\ldots\}\,|v\rangle_{123}$

    $|\widetilde{V}\rangle = \exp\{\ldots\}\,\delta_0\,|0\rangle_{123}$, with $\delta_0 = \prod_a (\sum_r \lambda_0^{a(r)})$

Two kinds of symmetry generators

Two kinds of symmetry generators

- Kinematical: Act at fixed x^+ (*i.e.* do preserve the light-cone conditions). Examples: $SO(4) \times SO(4)$ rotations, Q^+ ($\sim \theta_0 \, \mathrm{e}^{-i\mu\tau}$).

- Dynamical: All the other generators. Examples: $H = \frac{\partial}{\partial x^+}$ and
$$\bar{Q}_2^- = \int_0^{2\pi|\alpha|} d\sigma \left[\dot{X}^I \gamma^I \theta - e(\alpha) \partial_\sigma X^I \gamma^I \lambda + i\mu \frac{e(\alpha)}{2\pi\alpha'} X^I \gamma^I \Pi \theta \right] .$$
with $\theta = (\theta^1 + i\theta^2)/\sqrt{2}$ and $\lambda = e(\alpha)\bar{\theta}/(2\pi\alpha')$.

- Dynamical generators get corrections in the interacting theory. Examples: $\frac{\partial}{\partial x^+} = H = H_2 + g_s H_3 + \ldots$

Supersymmetry completion

- Consider the combinations

$$Q = \frac{1}{\sqrt{2}}\left(Q^- + \bar{Q}^-\right) \quad , \quad \widetilde{Q} = \frac{i}{\sqrt{2}}\left(Q^- - \bar{Q}^-\right) \ .$$

- From the anticommutations ($T = 0 \Leftrightarrow$ level-matching condition)

$$\{Q_{\dot{a}}, Q_{\dot{b}}\} = 2\delta_{\dot{a}\dot{b}}(H{+}T) \ , \ \{Q_{\dot{a}}, \widetilde{Q}_{\dot{b}}\} = i\mu\left[-(\gamma_{ij}\Pi)_{\dot{a}\dot{b}}J^{ij} + (\gamma_{i'j'}\Pi)_{\dot{a}\dot{b}}J^{i'j'}\right]$$

one finds at first order in g_s

$$\sum_{r=1}^{3} Q_2^{(r)\dot{a}}|Q_{3\dot{b}}\rangle + \sum_{r=1}^{3} Q_2^{(r)\dot{b}}|Q_{3\dot{a}}\rangle = 2\delta_{\dot{a}\dot{b}}|H_3\rangle$$

$$\sum_{r=1}^{3} Q_2^{(r)\dot{a}}|\widetilde{Q}_{3\dot{b}}\rangle + \sum_{r=1}^{3} \widetilde{Q}_2^{(r)\dot{b}}|Q_{3\dot{a}}\rangle = 0$$

Construct $|Q_{3\dot{a}}\rangle$ and $|H_3\rangle$ from $|V\rangle$ by adding for each case a suitable polynomial prefactor

- A first principle solution is still lacking... rely on some physical input

Completion A_s

- Ask for continuity of the $\mu \to 0$ limit (Spradlin and Volovich, Pankiewicz and Stefanski)

$$|H_3\rangle = \left(\tilde{K}^I\, K^J - \mu\,\kappa\,\delta^{IJ}\right)\, v_{IJ}(Y)\,|\widetilde{V}\rangle,$$

$$|Q_{3\,a}\rangle = \tilde{K}^I\, s^I_a(Y)\,|\widetilde{V}\rangle, \qquad |\tilde{Q}_{3\,a}\rangle = K^I\,\tilde{s}^I_a(Y)\,|\widetilde{V}\rangle$$

where K^I (Y^a) are object linear in the bosonic (fermionic) oscillators.

- $v_{IJ}(Y)$, $s^I_a(Y)$ and $\tilde{s}^I_a(Y)$ are intricate expressions

$$v_{IJ} = \delta_{IJ} - \tfrac{i}{2\kappa}(Y\Gamma_{IJ}Y) + \tfrac{1}{4!\,\kappa^2}(Y\Gamma_{IK}Y)(Y\Gamma_{JK}Y)$$

$$- \tfrac{1}{2\cdot 6!\,\kappa^3}(\Gamma_{IJ})_{ab}\,\epsilon^{ab}{}_{cdefgh}Y^c\ldots Y^h + \tfrac{1}{8!\,\kappa^4}\delta_{IJ}\,\epsilon_{abcdefgh}Y^a\ldots Y^h$$

- There is an equivalent form for this $|H_3\rangle$ built on $|V\rangle$ (Pankiewicz and Stefanski)

Historical remark

1) Last year it was hoped that the polynomial prefactor would reduce to the difference of ingoing and outgoing energies $(E_1 + E_2 - E_3)$

2) Various arguments/checks supporting this hope appeared (Constable et al., Huang and others)

3) However point 1) is not possible

   - Checking explicitly the failure of 1) is subtle (Pankiewicz)
   - However, the physical reason is simple: the continuity of $\mu \to 0$ limit forbids it! For $\mu = 0$ one has the holomorphic factorization and the prefactor has to be a product of a^1 and a^2, while $H_2 \sim a^{1\dagger} a^1 + a^{2\dagger} a^2$
   - In fact, in presence of just bosonic oscillators, $\mathcal{P} \sim \sum \frac{\omega_n}{\mu \alpha_r} a^{1\dagger}_{(r)n} a^2_{(r)n}$!

Completion B_s

- Give up the idea of smoothly connecting the PP-wave vertex to the one of flat space (Chu et al.)

- Then, there is a very simple solution (Di Vecchia et al.)

$$|H_3\rangle = \sum_{r=1}^{3} H_2^{(r)}|V\rangle \quad , \quad |Q_{3\dot{a}}\rangle = \sum_{r=1}^{3} Q_2^{(r)\dot{a}}\,|V\rangle \quad , \quad |\widetilde{Q}_{3\dot{a}}\rangle = \sum_{r=1}^{3} \widetilde{Q}_2^{(r)\dot{a}}\,|V\rangle$$

- It is consistent with the AdS/CFT expectation: amplitudes are proportional to the energy difference (Lee et al.)

- It allows to explain/use the arguments previously mentioned in point 2)

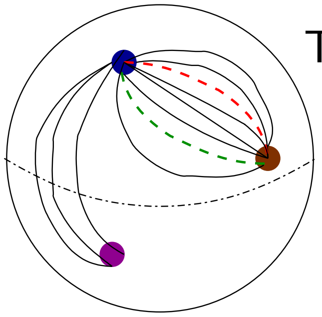
Tracing the string interaction on the GT side I

- First proposal (A_g): $\boxed{\mathbf{C_{3s}}/\sum_{\mathbf{r}} \mathbf{H_2^{(r)}} = \mathbf{C_{3g}}}$ (Constable et al., Di Vecchia et al.,...)

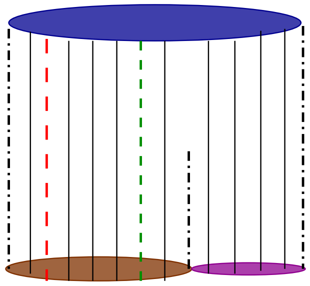
$$C_{3g} = \lim_{x_3 \to \infty} \langle \bar{O}^{J_3}_{ij,n}(x_3) O^{J_2}_{ij,m}(x_2) O^{J_1}_{vac}(x_1) \rangle$$

where $\bar{O}(x)$ is defined with an inversion $\bar{O}(x) = O^*(1/x)$

- This proposal works only with the completion B for the 3-string interaction: $C_{3g} = (\langle 1| \otimes \langle 2| \otimes \langle 3|) \, |V\rangle_{planar}$
- At leading order in λ' there is a nice physical interpretation



The GT correlator in the free limit $(\lambda' \to 0)$
reconstruct the 3-string diagram, where
$\longleftrightarrow$
the single bits are free (ultralocal interaction!)
In fact, when $\mu \to \infty$, then $\mathcal{L}_b \sim X^2$



Explicit checks

- Consider the following states dual to 2-impurity operators:

$$|bos_n\rangle_s = \hat{a}_n^{\phi\dagger}\hat{a}_{-n}^{\phi\dagger}|v\rangle_\alpha \sim \sum_{l=0}^{J}\mathrm{Tr}[\phi Z^l \phi Z^{(J-l)}]\mathrm{e}^{2\pi\mathrm{i}\frac{(l+1)n}{J+2}} + \dots$$

$$|fer_n\rangle_s = \hat{\theta}_n^{a\dagger}\hat{\theta}_{-n}^{a\dagger}|v\rangle_\alpha \sim \sum_{l=0}^{J}\mathrm{Tr}[\lambda^a Z^l \lambda^a Z^{(J-l)}]\mathrm{e}^{2\pi\mathrm{i}\frac{(l+1)n}{J+2}} + \dots$$

- Compute some amplitudes with these states by using the kinematical vertex $|V\rangle$. In particular we are interested in the $\mu \to \infty$ limit

1) $\left({}^3_s\langle bos_n| \otimes {}^2_s\langle bos_m| \otimes {}_{\alpha_1}\langle v|\right)|V\rangle \sim \frac{y}{\pi^2}\sin^2(\pi n y)\left[\frac{1}{(m-ny)^2} + \frac{1}{(m+ny)^2}\right]$

2) $\left({}^3_s\langle fer_n| \otimes {}^2_s\langle fer_m| \otimes {}_{\alpha_1}\langle v|\right)|V\rangle \sim \frac{y}{\pi^2}\sin^2(\pi n y)\left[\frac{1}{(m-ny)^2} - \frac{1}{(m+ny)^2}\right]$

where $y = J_2/J_3 = |\alpha_2/\alpha_3|$

Summary of approach A_g

$+$ Many checks with all kinds of impurities. This means that the whole vertex is tested

$+$ Clear pictorial understanding of the string/GT duality

$\pm$ Conceptually very close to the usual interpretation of AdS/CFT duality

? The role of operator mixing on the string side

? Space-time dependence of the correlators

?? Subleading corrections in λ' (Can one really reconstruct the world-sheet lagrangian from GT quantum expansion ?)

Tracing the string interaction on the GT side II

- Second proposal (A_g): $\boxed{\mathbf{\Delta} - \mathbf{J} = \mathbf{H}/\mu}$ is an exact operator relation including the correction due to the interaction: $H_3, H_4, \ldots$(Gross et al.)
- Interpret multi-string state as multi-trace operators
- Study only 2-point function to extract the anomalous dimension

$$\langle \bar{O}_\alpha(x)\, O_\beta(0)\rangle = \left(\frac{g_{YM}^2}{8\pi^2|x|^2}\right)^{J+2} \left(S_{\alpha\beta} + T_{\alpha\beta}\log|x\,\Lambda|^{-2}\right)$$

- $S_{\alpha\beta}$ induces a scalar product among the O's and $T_{\alpha\beta}$ is connected to the full string hamiltonian
- This approach is less constrained than the first one

Basis dependent checks

- Introduce a basis where $S_{\alpha\beta}$ is diagonal (?)...

$$\tilde{O}^J_{12,m} = O^J_{12,m} - \sum_{k,r} \frac{g_2\, r^{3/2}\sqrt{1-r}\,\sin^2(\pi m r)}{2\sqrt{J}\,\pi^2(k-mr)^2}\, O^{J'}_{12,k}\mathrm{Tr}[Z^{J-J'}] + \ldots$$

- ...many other different possibilities!. This choice is priviledged, because $\widetilde{T}_{\alpha\beta}$ is equal to C_{3s} computed with the "smooth" vertex A_s (Gross et al. v2)
- Of course one can chose a different basis $\hat{O}$ so that $\widehat{T}_{\alpha\beta}$ is equal to C_{3s} computed with the vertex B_s (Gross et al. v1).
- Many checks (with the basis $\widetilde{T}_{\alpha\beta}$) for bosonic impurities (Gomis et al. v2, . . .)

Basis independent checks

- Consider the light-cone string Hamiltonian $H = H_2 + g_2 H_3 + g_2^2 H_4$ and diagonalize it perturbatively in g_2 starting from "free" eigenvectors of H_2 (very reminiscent of what done on the GT side... same problem with non-degenerate perturbation theory).

- H_4 is derived from the susy algebra: $2H_4 = \{Q_3, Q_3\} + \dots$

- Let $|n\rangle$ be a state with 2 bosonic impuirities of level n

$$\delta E^{(2)}_{|n\rangle} = {g_2}^2 \sum_{|\alpha\rangle} \Big\{ \frac{\langle n|\widehat{H}_3|\alpha\rangle \, \langle \alpha|\widehat{H}_3|n\rangle}{E^{(0)}_n - E^{(0)}_\alpha} + \frac{1}{8} \langle n|Q_{3\,a}|\alpha\rangle \, \langle \alpha|Q_{3\,a}|n\rangle \Big\} \, .$$

- If the sum on $|\alpha\rangle$ is truncated to the 2 oscillators subspace, then $\delta E^{(2)}_{|n\rangle} = \lambda' g_2^2 \left(\frac{1}{12} + \frac{35}{32\pi^2 m^2}\right)$ for all states $|n\rangle$ with 2 bosonic oscillators

Summary of approach B_g

\+ General applicability: in principle it's valid to all order in g_2 and λ'

± Natural generalization of the BMN analysis. However, far from the usual interpretation of AdS/CFT duality

? Basis-dep.: the first check at subleading order in λ' show a numerical disagreement

? Basis-indep.: no real string computation. The truncation to impurity preserving sector is puzzling

?? Checks limited to the bosonic sector. Only a small part of the string vertex is tested.

Conclusions

- The PP-wave background is the best laboratory we have to test the AdS/CFT duality at the string level (i.e. beyond supergravity)
- It helped to understand the importance building a dictionary between the string and the gauge theory side (solving string theory on $AdS^5 \times S^5$ is not enough!).
- However, the relation between the BMN and the AdS/CFT is not completely clear
- What developments ?
  - Hopefully it will provide a severe test for a few conjectures inspired by the AdS/CFT duality in the last years
  - Test/inspire the analysis of the study of string dynamics on $AdS^5 \times S^5$ and $\mathcal{N} = 4$ SYM theory (talks by Tseytlin and Russo J.)

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